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Trace Formula for Orthogonal Polynomials with Asymptotically 2-Periodic Recurrence Coefficients

Boris P. Osilenker

Department of Mathematics, Moscow State Civil Engineering University, 2-oj Krestovskii pereulok Dom 4, Apt. 55, 129041 Moscow, Russia

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1. INTRODUCTION

Suppose μ is a positive probability measure on a compact set on the real line, that is, μ is a positive Borel measure with $\int d\mu = 1$. Then there is a unique sequence of polynomials

$$p_n(x) = k_n x^n + \cdots, \qquad k_n > 0 \quad (n \in \mathbb{Z}_+ = \{0, 1, ...\})$$

such that

$$\int p_m(x)\, p_n(x)\, d\mu(x) = \delta_{mn} \qquad (m, n \in \mathbb{Z}_+).$$

These orthonormal polynomials satisfy a three-term recurrence relation

$$xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) \qquad (n \in \mathbb{Z}_+) \tag{1.1}$$

with the initial conditions

$$p_{-1}(x) = 0, \qquad p_0(x) = 1, \qquad a_0 = 0, \tag{1.2}$$

where $a_{n+1} = k_n / k_{n+1} > 0$ and $b_n \in \mathbb{R}$.

Conversely, by Favard's Theorem, if the polynomials $p_n(x)$ are given by the recurrence formula (1.1) with $a_{n+1} > 0$ and $b_n \in \mathbb{R}$, then there exists a positive Boreal measure μ such that $\{p_n\}$ $(n \in \mathbb{Z}_+)$ is an orthonormal polynomial system with respect to the measure μ. If a_n and b_n are bounded, then the measure μ is unique and the support of μ is compact.

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The following result (Trace Formula) establishes the connection between Jacobi matrices

$$J=\begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and their spectral measures.

THEOREM 1 [8, 14] (see also [16]). *If* $\operatorname{Supp}(\mu)=[-1,1]$ *and if the recursion coefficients* $\{a_{n+1}\}$ *and* $\{b_n\}$ *satisfy*

$$\lim_{n\to\infty} a_n=\tfrac{1}{2}, \qquad \lim_{n\to\infty} b_n=0 \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} (|a_{n+1}-a_n|+|b_{n+1}-b_n|)<\infty, \tag{1.4}$$

then

$$\sum_{n=0}^{\infty} [(a_{n+1}^2-a_n^2)\, p_n^2(x)+a_n(b_n-b_{n-1})\, p_{n-1}(x)\, p_n(x)]=\frac{1}{2\pi}\frac{\sqrt{1-x^2}}{\mu'(x)}$$

holds uniformly on all compact sets in $(-1,1)$. *In addition, the measure* μ *is absolutely continuous in the open interval* $(-1,1)$, $\mu'(x)>0$ *for all* $x\in(-1,1)$, *and* $\mu'(x)$ *is continuous in* $(-1,1)$.

Given two periodic sequences $\{a_{n+1}^0\}$ $(a_{n+1}^0>0)$ *and* $\{b_n^0\}$ $(b_n^0\in\mathbb{R})$ $(n\in\mathbb{Z}_+)$ *with period* $N\geqslant 1$, *the Jacobi matrix* J *is called asymptotically N-periodic* $(J\in AP_N)$, *if*

$$\lim_{n\to\infty} [|a_n-a_n^0|+|b_n-b_n^0|]=0. \tag{1.5}$$

In a survey [16] P. Nevai has posed the following problem: Extend the Trace Formula to asymptotically N-periodic Jacobi matrices. We investigate this problem for the case $N=2$.

2. REPRESENTATIONS OF THE KERNELS

We assume that two periodic sequences $a_{n+1}^0>0$ and b_n^0 $(n\in\mathbb{Z}_+)$ are given such that

$$a_{n+2}^0=a_n^0 \quad (n=1,2,...), \qquad b_{n+2}^0=b_n^0 \quad (n=0,1,2,...)$$

(i.e., period $N=2$), and that the recurrence coefficients a_{n+1} and b_n satisfy (1.5). We will write $J \in AP_2$.

Denote the orthonormal polynomials with periodic recurrence coefficients a^0_{n+1} and b^0_n by $q_n(x)$. Then

$$xq_n(x) = a^0_{n+1} q_{n+1}(x) + b^0_n q_n(x) + a^0_n q_{n-1}(x) \qquad (n \in \mathbb{Z}_+)$$
$$q_{-1}(x) = 0, \quad q_0(x) = 1.$$

Let

$$T(x) = \frac{1}{2}\left[q_2(x) - \frac{a^0_2}{a^0_1} \right]. \tag{2.1}$$

The essential spectrum of the polynomials $q_n(x)$, resp. $p_n(x)$, consists of two intervals E, where $E = \{x \in \mathbb{R}, -1 \leqslant T(x) \leqslant 1\}$. The set E is of the form

$$\left[\alpha, \frac{\alpha+\beta}{2} - \frac{\beta-\alpha}{2} t\right] \cup \left[\frac{\alpha+\beta}{2} + t\frac{\beta-\alpha}{2}, \beta\right] \qquad \text{for some } 0 \leqslant t < 1,$$

or in other words, if -1, 1 is the smallest and largest boundary point of E (this can be obtained easily by a linear transformation of $q_n(x)$ resp. $p_n(x)$) then E is of the form

$$E = [-1, -t] \cup [t, 1] \qquad \text{for some } 0 \leqslant t < 1.$$

These facts follow from [20, 21]. The special case when the intervals touch each other (the set $\{T^2(x) = 1\}$ consists of the endpoints of the intervals) leads to sieved orthogonal polynomials (see details in [1, 9–11, 21]).

LEMMA 1. *If $J \in AP_2$, then the following recurrence relation is valid,*

$$s(x)\, p_n(x) = \alpha_{n+2} p_{n+2}(x) + \beta_{n+1} p_{n+1}(x) + \gamma_n p_n(x) + \beta_n p_{n-1}(x) + \alpha_n p_{n-2}(x) \qquad (n \in \mathbb{Z}_+) \tag{2.2}$$

with boundary conditions

$$p_{-2}(x) = 0, \qquad p_{-1}(x) = 0, \qquad \alpha_0 = \alpha_1 = 0, \qquad \beta_0 = 0, \tag{2.3}$$

where

$$\begin{gathered} s(x) = 2a^0_1 a^0_2 T(x) = (x - b^0_0)(x - b^0_1) - (a^0_1)^2 - (a^0_2)^2 \\ \alpha_{n+2} = a_{n+1} a_{n+2}, \qquad \beta_n = a_n(b_{n-1} + b_n - b^0_0 - b^0_1) \\ \gamma_n = a^2_n + a^2_{n+1} - (a^0_1)^2 - (a^0_2)^2 + (b_n - b^0_0)(b_n - b^0_1) \end{gathered} \tag{2.4}$$

with

$$\lim_{n\to\infty} \alpha_n = a_1^0 a_2^0, \qquad \lim_{n\to\infty} \beta_n = 0, \qquad \lim_{n\to\infty} \gamma_n = 0. \tag{2.5}$$

In fact, Lemma 1 follows from the definition of AP_2 and the five-term recurrence relation

$$\begin{aligned} x^2 p_n(x) = {} & a_{n+1} a_{n+2} p_{n+2}(x) + a_{n+1}(b_n + b_{n+1})\, p_{n+1}(x) \\ & + (a_n^2 + a_{n+1}^2 + b_n^2)\, p_n(x) + a_n(b_{n-1} + b_n)\, p_{n-1}(x) \\ & + a_n a_{n-1} p_{n-2}(x) \end{aligned}$$

by a direct computation.

LEMMA 2. *Let $J \in AP_2$. Then for the Dirichlet kernel*

$$D_n(t, x) = \sum_{k=0}^{n} p_k(t)\, p_k(x)$$

the representation

$$\begin{aligned} [s(t) - s(x)]\, D_n(t, x) = {} & \alpha_{n+2}[p_{n+2}(t)\, p_n(x) - p_n(t)\, p_{n+2}(x)] \\ & + \alpha_{n+1}[p_{n+1}(t)\, p_{n-1}(x) - p_{n-1}(t)\, p_{n+1}(x)] \\ & + \beta_{n+1}[p_{n+1}(t)\, p_n(x) - p_n(t)\, p_{n+1}(x)] \end{aligned} \tag{2.6}$$

holds. Here $s(x)$, α_n, β_n are defined by (2.4).

In fact, formula (2.6) follows from (2.2) and (2.3) by straightforward calculations.

LEMMA 3. *If $J \in AP_2$, then the following representation of the Fejér kernel,*

$$F_n(t, x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t, x),$$

is valid

$$\begin{aligned} & (n+1)[s(t) - s(x)]^2\, F_n(t, x) \\ & \quad = \xi_n(t, x) + \xi_n(x, t) + G_n(t, x) + G_n(x, t), \end{aligned} \tag{2.7}$$

where

$$
\begin{aligned}
G_n(t,x) = {} & \alpha_{n+2}^2[p_n(t)\,p_n(x) - p_{n+2}(t)\,p_{n+2}(x)] \\
& - 2[\alpha_n^2 p_n(t)\,p_n(x) + \alpha_{n+1}^2 p_{n+1}(t) p_{n+1}(x)] \\
& + \alpha_{n+2}\alpha_{n+4} p_{n+4}(t)\,p_n(x) + 2\alpha_{n+1}\alpha_{n+3} p_{n+3}(t)\,p_{n-1}(x) \\
& + \alpha_n\alpha_{n+2} p_{n+2}(t)\,p_{n-2}(x) + \alpha_{n+2}(\gamma_{n+2} - \gamma_n)\,p_{n+2}(t)\,p_n(x) \\
& - \beta_{n+1}^2 p_{n+1}(t)\,p_{n+1}(x) + (\alpha_{n+2}\beta_{n+3} + \alpha_{n+3}\beta_{n+1})\,p_{n+3}(t)\,p_n(x) \\
& + 2\alpha_{n+1}\beta_{n+2} p_{n+2}(t)\,p_{n-1}(x) + \beta_{n+1}\beta_{n+2} p_{n+2}(t)\,p_n(x) \\
& + (\alpha_{n+2}\beta_{n+2} - 2\alpha_{n+1}\beta_n)\,p_{n+1}(t)\,p_n(x) \\
& - \alpha_{n+2}\beta_{n+1} p_{n+2}(t)\,p_{n+1}(x) - \alpha_{n+2}\beta_{n+1} p_{n+1}(t)\,p_{n+2}(x) \\
& - \alpha_{n+1}\beta_n p_n(t)\,p_{n+1}(x) \qquad (2.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_n(t,x) = {} & 2\sum_{k=0}^{n-1} (\alpha_{k+2}^2 - \alpha_k^2)\,p_k(t)\,p_k(x) + \sum_{k=0}^{n} (\beta_{k+1}^2 - \beta_k^2)\,p_k(t)\,p_k(x) \\
& + 2\sum_{k=0}^{n-1} \alpha_{k+2}(\gamma_{k+2} - \gamma_k)\,p_{k+2}(t)\,p_k(x) \\
& + \sum_{k=0}^{n-2} (\alpha_{k+2}\beta_{k+3} - \alpha_{k+3}\beta_{k+1})\,p_{k+3}(t)\,p_k(x) \\
& + 2\sum_{k=0}^{n-1} (\alpha_{k+2}\beta_{k+2} - \alpha_{k+1}\beta_k)\,p_{k+1}(t)\,p_k(x) \\
& + \sum_{k=0}^{n} \beta_{k+1}(\gamma_{k+1} - \gamma_k)\,p_{k+1}(t)\,p_k(x) \\
& + \sum_{k=0}^{n-1} (\alpha_{k+2}\beta_{k+2} - \alpha_{k+1}\beta_k)\,p_k(t)\,p_{k+1}(x), \qquad (2.9)
\end{aligned}
$$

where α_n, β_n, γ_n are defined by (2.4).

Proof. By formula (2.6) one has

$$
(n+1)[s(t) - s(x)]^2 F_n(t,x) = \zeta_n(t,x) + \zeta_n(x,t), \qquad (2.10)
$$

where

$$
\zeta_n(t,x) = \sum_{k=0}^{n} \theta_k(t,x) \qquad (2.11)
$$

with

$$\theta_k(t,x) = \alpha_{k+2} s(t)\, p_{k+2}(t)\, p_k(x) + \alpha_{k+1} s(t)\, p_{k+1}(t)\, p_{k-1}(x)$$
$$+ \beta_{k+1} s(t)\, p_{k+1}(t)\, p_k(x) - \alpha_{k+2} p_{k+2}(t)\, s(x)\, p_k(x)$$
$$- \alpha_{k+1} p_{k+1}(t)\, s(x)\, p_{k-1}(x) - \beta_{k+1} p_{k+1}(t)\, s(x)\, p_k(x).$$

The recurrence relation (2.2) yields

$$\theta_k(t,x) = \alpha_{k+2}\alpha_{k+4} p_{k+4}(t)\, p_k(x) + \alpha_{k+2}\beta_{k+3} p_{k+3}(t)\, p_k(x)$$
$$+ \alpha_{k+2}\gamma_{k+2} p_{k+2}(t)\, p_k(x) + \alpha_{k+2}\beta_{k+2} p_{k+1}(t)\, p_k(x)$$
$$+ \alpha_{k+2}^2 p_k(t)\, p_k(x) + \alpha_{k+1}\alpha_{k+3} p_{k+3}(t)\, p_{k-1}(x)$$
$$+ \alpha_{k+1}\beta_{k+2} p_{k+2}(t)\, p_{k-1}(x) + \alpha_{k+1}\gamma_{k+1} p_{k+1}(t)\, p_{k-1}(x)$$
$$+ \alpha_{k+1}\beta_{k+1} p_k(t)\, p_{k-1}(x) + \alpha_{k+3}\beta_{k+1} p_{k+3}(t)\, p_k(x)$$
$$+ \beta_{k+1}\beta_{k+2} p_{k+2}(t)\, p_k(x) + \beta_{k+1}\gamma_{k+1} p_{k+1}(t)\, p_k(x)$$
$$+ \beta_{k+1}^2 p_k(t)\, p_k(x) + \alpha_{k+1}\beta_{k+1} p_{k-1}(t)\, p_k(x)$$
$$- \alpha_{k+2}^2 p_{k+2}(t)\, p_{k+2}(x) - \alpha_{k+2}\beta_{k+1} p_{k+2}(t)\, p_{k+1}(x)$$
$$- \alpha_{k+2}\gamma_k p_{k+2}(t)\, p_k(x) - \alpha_{k+2}\beta_k p_{k+2}(t)\, p_{k-1}(x)$$
$$- \alpha_{k+2}\beta_{k+1} p_{k+1}(t)\, p_{k+2}(x) - \beta_{k+1}^2 p_{k+1}(t)\, p_{k+1}(x)$$
$$- \beta_{k+1}\gamma_k p_{k+1}(t)\, p_k(x) - \alpha_{k+2}\alpha_k p_{k+2}(t)\, p_{k-2}(x)$$
$$- \alpha_{k+1}^2 p_{k+1}(t)\, p_{k+1}(x) - \alpha_{k-1}\alpha_{k+1} p_{k+1}(t)\, p_{k-3}(x)$$
$$- \beta_{k+1}\beta_k p_{k+1}(t)\, p_{k-1}(x) - \alpha_k\beta_{k+1} p_{k+1}(t)\, p_{k-2}(x)$$
$$- \alpha_{k+1}\beta_k p_{k+1}(t)\, p_k(x) - \alpha_{k+1}\gamma_{k-1} p_{k+1}(t)\, p_{k-1}(x)$$
$$- \alpha_{k+1}\beta_{k+1} p_{k+1}(t)\, p_{k-2}(x) + \alpha_{k+1}^2 p_{k-1}(t)\, p_{k-1}(x).$$

For the calculation of $\zeta_n(t,x)$ we regroup similar terms of the last relation. Then $\zeta_n(t,x)$ can be representated in the form

$$\zeta_n(t,x) = \sum_{i=1}^{6} \sigma_i(t,x), \tag{2.12}$$

where

$$\sigma_1(t,x)=\sum_{k=0}^{n}\alpha_{k+2}^2 p_k(t)\,p_k(x)+\sum_{k=0}^{n}\alpha_{k+1}^2 p_{k-1}(t)\,p_{k-1}(x)$$

$$-\sum_{k=0}^{n}\alpha_{k+2}^2 p_{k+2}(t)\,p_{k+2}(x)-\sum_{k=0}^{n}\alpha_{k+1}^2 p_{k+1}(t)\,p_{k+1}(x)$$

$$+\sum_{k=0}^{n}\beta_{k+1}^2 p_k(t)\,p_k(x)-\sum_{k=0}^{n}\beta_{k+1}^2 p_{k+1}(t)\,p_{k+1}(x);$$

$$\sigma_2(t,x)=\sum_{k=0}^{n}\alpha_{k+2}\alpha_{k+4} p_{k+4}(t)\,p_k(x)-\sum_{k=0}^{n}\alpha_k\alpha_{k+2} p_{k+2}(t)\,p_{k-2}(x)$$

$$+\sum_{k=0}^{n}\alpha_{k+1}\alpha_{k+3} p_{k+3}(t)\,p_{k-1}(x)-\sum_{k=0}^{n}\alpha_{k-1}\alpha_k p_{k+1}(t)\,p_{k-3}(x);$$

$$\sigma_3(t,x)=\sum_{k=0}^{n}\alpha_{k+2}\beta_{k+3} p_{k+3}(t)\,p_k(x)-\sum_{k=0}^{n}\alpha_{k+2}\beta_k p_{k+2}(t)\,p_{k-1}(x)$$

$$+\sum_{k=0}^{n}\alpha_{k+1}\beta_{k+2} p_{k+2}(t)\,p_{k-1}(x)-\sum_{k=0}^{n}\alpha_{k+1}\beta_{k-1} p_{k+1}(t)\,p_{k-2}(x)$$

$$+\sum_{k=0}^{n}\alpha_{k+3}\beta_{k+1} p_{k+3}(t)\,p_k(x)-\sum_{k=0}^{n}\alpha_k\beta_{k+1} p_{k+1}(t)\,p_{k-2}(x);$$

$$\sigma_4(t,x)=\sum_{k=0}^{n}\alpha_{k+2}\gamma_{k+2} p_{k+2}(t)\,p_k(x)+\sum_{k=0}^{n}\alpha_{k+1}\gamma_{k+1} p_{k+1}(t)\,p_{k-1}(x)$$

$$+\sum_{k=0}^{n}\beta_{k+1}\beta_{k+2} p_{k+2}(t)\,p_k(x)-\sum_{k=0}^{n}\alpha_{k+2}\gamma_k p_{k+2}(t)\,p_k(x)$$

$$-\sum_{k=0}^{n}\beta_k\beta_{k+1} p_{k+1}(t)\,p_{k-1}(x)-\sum_{k=0}^{n}\alpha_{k+1}\gamma_{k-1} p_{k+1}(t)\,p_{k-1}(x);$$

$$\sigma_5(t,x)=\sum_{k=0}^{n}\alpha_{k+2}\beta_{k+2} p_{k+1}(t)\,p_k(x)+\sum_{k=0}^{n}\alpha_{k+1}\beta_{k+1} p_k(t)\,p_{k-1}(x)$$

$$+\sum_{k=0}^{n}\beta_{k+1}\gamma_{k+1} p_{k+1}(t)\,p_k(x)-\sum_{k=0}^{n}\alpha_{k+2}\beta_{k+1} p_{k+2}(t)\,p_{k+1}(x)$$

$$-\sum_{k=0}^{n}\beta_{k+1}\gamma_k p_{k+1}(t)\,p_k(x)-\sum_{k=0}^{n}\alpha_{k+1}\beta_k p_{k+1}(t)\,p_k(x);$$

$$\sigma_6(t,x)=\sum_{k=0}^{n}\alpha_{k+1}\beta_{k+1} p_{k-1}(t)\,p_k(x)-\sum_{k=0}^{n}\alpha_{k+2}\beta_{k+1} p_{k+1}(t)\,p_{k+2}(x).$$

Using Abel's summation by parts and the initial conditions (2.3), it is not difficult to show that the following formulas are valid

$$\sigma_1(t,x)=\alpha_{n+2}^2[p_n(t)\,p_n(x)-p_{n+2}(t)\,p_{n+2}(x)]$$
$$-2[\alpha_n^2 p_n(t)\,p_n(x)+\alpha_{n+1}^2 p_{n+1}(t)\,p_{n+1}(x)]$$
$$-\beta_{n+1}^2 p_{n+1}(t)\,p_{n+1}(x)+2\sum_{k=0}^{n-1}(\alpha_{k+2}^2-\alpha_k^2)\,p_k(t)\,p_k(x)$$
$$+\sum_{k=0}^{n}(\beta_{k+1}^2-\beta_k^2)\,p_k(t)\,p_k(x);$$

$$\sigma_2(t,x)=\alpha_{n+2}\alpha_{n+4}p_{n+4}(t)\,p_n(x)+2\alpha_{n+1}\alpha_{n+3}p_{n+3}(t)\,p_{n-1}(x)$$
$$+\alpha_n\alpha_{n+2}p_{n+2}(t)\,p_{n-2}(x);$$

$$\sigma_3(t,x)=(\alpha_{n+2}\beta_{n+3}+\alpha_{n+3}\beta_{n+1})\,p_{n+3}(t)\,p_n(x)$$
$$+2\alpha_{n+1}\beta_{n+2}p_{n+2}(t)\,p_{n-1}(x)+\sum_{k=0}^{n-2}[\alpha_{k+2}(\beta_{k+3}-\beta_{k+1})$$
$$+(\alpha_{k+2}-\alpha_{k+3})\,\beta_{k+1}]\,p_{k+3}(t)\,p_k(x);$$

$$\sigma_4(t,x)=\alpha_{n+2}(\gamma_{n+2}-\gamma_n)\,p_{n+2}(t)\,p_n(x)+\beta_{n+1}\beta_{n+2}p_{n+2}(t)\,p_n(x)$$
$$+2\sum_{k=0}^{n-1}\alpha_{k+2}(\gamma_{k+2}-\gamma_k)\,p_{k+2}(t)\,p_k(x);$$

$$\sigma_5(t,x)=(\alpha_{n+2}\beta_{n+2}-2\alpha_{n+1}\beta_n)\,p_{n+1}(t)\,p_n(x)$$
$$-\alpha_{n+2}\beta_{n+1}p_{n+2}(t)\,p_{n+1}(x)+2\sum_{k=0}^{n-1}[(\alpha_{k+2}-\alpha_{k+1})\,\beta_{k+2}$$
$$+(\beta_{k+2}-\beta_k)\,\alpha_{k+1}]\,p_{k+1}(t)\,p_k(x)$$
$$+\sum_{k=0}^{n}\beta_{k+1}(\gamma_{k+1}-\gamma_k)\,p_{k+1}(t)\,p_k(x);$$

$$\sigma_6(t,x)=-\alpha_{n+2}\beta_{n+1}p_{n+1}(t)\,p_{n+2}(x)-\alpha_{n+1}\beta_n p_n(t)\,p_{n+1}(x)$$
$$+\sum_{k=0}^{n-1}[\alpha_{k+2}(\beta_{k+2}-\beta_k)+(\alpha_{k+2}-\alpha_{k+1})\,\beta_k]\,p_k(t)\,p_{k+1}(x).$$

The representation (2.7)–(2.9) follows from (2.10)–(2.12) and the last six relations. Lemma 3 is completely proved.

Remark. For $N=1$ the representation of $F_n(t,x)$ was given in [17, 18] (see also [19] for $N=2$). Fejér's kernel plays an important role in some problems of summability of Fourier series in orthogonal polynomials [17, 18].

The following assertion can be inferred from Lemma 3, if we put $t=x$.

COROLLARY 1. *If $J \in AP_2$, then*

$$2 \sum_{k=0}^{n-1} (\alpha_{k+2}^2 - \alpha_k^2)\, p_k^2(x) + \sum_{k=0}^{n} (\beta_{k+1}^2 - \beta_k^2)\, p_k^2(x)$$
$$+ 2 \sum_{k=0}^{n-1} \alpha_{k+2}(\gamma_{k+2} - \gamma_k)\, p_k(x)\, p_{k+2}(x)$$
$$+ 3 \sum_{k=0}^{n-1} (\alpha_{k+2}\beta_{k+2} - \alpha_{k+1}\beta_k)\, p_k(x)\, p_{k+1}(x)$$
$$+ \sum_{k=0}^{n-2} (\alpha_{k+2}\beta_{k+3} - \alpha_{k+3}\beta_{k+1})\, p_k(x)\, p_{k+3}(x)$$
$$+ \sum_{k=0}^{n} \beta_{k+1}(\gamma_{k+1} - \gamma_k)\, p_k(x)\, p_{k+1}(x) = G_n(x),$$

where

$$\begin{aligned} G_n(x) = {} & (2\alpha_n^2 - \alpha_{n+2}^2)\, p_n^2(x) + (2\alpha_{n+1}^2 + \beta_{n+1}^2)\, p_{n+1}^2(x) + \alpha_{n+2}^2 p_{n+2}^2(x) \\ & - \alpha_{n+2}\alpha_{n+4} p_n(x)\, p_{n+4}(x) - 2\alpha_{n+1}\alpha_{n+3} p_{n-1}(x)\, p_{n+3}(x) \\ & - \alpha_n\alpha_{n+2} p_{n-2}(x)\, p_{n+2}(x) - \alpha_{n+2}(\gamma_{n+2} - \gamma_n)\, p_n(x)\, p_{n+2}(x) \\ & - \beta_{n+1}\beta_{n+2} p_n(x)\, p_{n+2}(x) - 2\alpha_{n+1}\beta_{n+2} p_{n-1}(x)\, p_{n+2}(x) \\ & - (\alpha_{n+2}\beta_{n+3} + \alpha_{n+3}\beta_{n+1})\, p_n(x)\, p_{n+3}(x) \\ & + 2\alpha_{n+2}\beta_{n+1} p_{n+1}(x)\, p_{n+2}(x) \\ & + (3\alpha_{n+1}\beta_n - \alpha_{n+2}\beta_{n+2})\, p_n(x)\, p_{n+1}(x), \end{aligned} \tag{2.13}$$

where α_n, β_n, and γ_n are defined by (2.4).

Remark. For $N = 1$ formula (2.13) was obtained by J. Dombrowski [7].

3. TRACE FORMULA

If $\{p_n\}$ is a system of orthonormal polynomials, satisfying (1.1), (1.2), then one can introduce the associated polynomials $\{p_n^{(m)}(x)\}$ of order m ($m \in \mathbb{Z}_+$) by the shifted recurrence formula

$$xp_n^{(m)}(x) = a_{n+m+1} p_{n+1}^{(m)}(x) + b_{n+m} p_n^{(m)}(x) + a_{n+m} p_{n-1}^{(m)}(x) \qquad (n \in \mathbb{Z}) \tag{3.1}$$

with boundary conditions

$$p_{-1}^{(m)}(x) = 0, \qquad p_0^{(m)}(x) = 1. \tag{3.2}$$

We need the following result

LEMMA 4 [11]. *Assume that the Jacobi matrix $J \in AP_2$. Then for every continuous function f and for all integers k and j one has*

$$\lim_{n\to\infty} \int f(x)\, p_{2n+j}(x)\, p_{2n+k}(x)\, d\mu(x)$$
$$= \frac{1}{4\pi a^0_{j+1} a^0_{k+1}} \int_E \frac{f(x)\, \mathrm{Sign}[T'(x)]}{\sqrt{1-T^2(x)}}$$
$$\times [a^0_{k+1} q^{(j+1)}_{k-j+1}(x) + a^0_{j+1} q^{(k+1)}_{j-k+1}(x)]\, dx, \tag{3.3}$$

where

$$a^0_{m+k+1} q^{(m)}_k(x) = -a^0_m q^{(m+k+1)}_{-k-2}(x) \qquad (k<0). \tag{3.4}$$

The next assertion gives the weak type asymptotics.

LEMMA 5. *Suppose that the Jacobi matrix $J \in AP_2$. Then for every continuous function f and for all integers k one has*

$$\lim_{n\to\infty} \int_E f(x)\, G_{2n+k}(x)\, d\mu(x)$$
$$= \frac{4(a^0_1 a^0_2)^2}{\pi} \int_E f(x)\, |T'(x)|\, \sqrt{1-T^2(x)}\, dx$$
$$= \frac{1}{\pi} \int_E f(x)\, |2x - b^0_k - b^0_{k+1}|$$
$$\times \sqrt{\begin{matrix}[(a^0_{k+2}+a^0_{k+1})^2 - (x-b^0_k)(x-b^0_{k+1})] \times \\ [(x-b^0_k)(x-b^0_{k+1}) - (a^0_{k+2}-a^0_{k+1})^2]\end{matrix}}\, dx. \tag{3.5}$$

Proof. Using the definition of the class AP_2 and (2.5), we have

$$I^{(2)}_k := \lim_{n\to\infty} \int_E f(x)\, G_{2n+k}(x)\, d\mu(x)$$
$$= \lim_{n\to\infty} \int_E f(x) \{ \alpha^2_{2n+k+2} p^2_{2n+k+2}(x) + 2\alpha^2_{2n+k} p^2_{2n+k}(x)$$
$$- \alpha^2_{2n+k+2} p^2_{2n+k}(x) + 2\alpha^2_{2n+k+1} p^2_{2n+k+1}(x)$$
$$- \alpha_{2n+k+2}\alpha_{2n+k+4} p_{2n+k}(x)\, p_{2n+k+4}(x)$$
$$- 2\alpha_{2n+k+1}\alpha_{2n+k+3} p_{2n+k-1}(x)\, p_{2n+k+3}(x)$$
$$- \alpha_{2n+k}\alpha_{2n+k+2} p_{2n+k-2}(x)\, p_{2n+k+2}(x)\}\, d\mu(x).$$

By (1.5), (3.3), and (3.4)

$$I_k^{(2)} = (a_{k+1}^0 a_{k+2}^0)^2 \frac{1}{4\pi} \int_E \frac{f(x)\,\mathrm{Sign}[T'(x)]}{\sqrt{1-T^2(x)}}$$

$$\times \left\{ \frac{2}{a_{k+1}^0} q_1^{(k+3)}(x) + \frac{4}{a_k^0} q_1^{(k+2)}(x) + \frac{2}{a_{k+1}^0} q_1^{(k+1)}(x) \right.$$

$$- \frac{1}{a_{k+1}^0} [q_5^{(k+1)}(x) + q_{-3}^{(k+5)}(x)] - \frac{2}{a_k^0} [q_5^{(k)}(x) + q_{-3}^{(k+4)}(x)]$$

$$\left. - \frac{1}{a_{k+1}^0} [q_5^{(k-1)}(x) + q_{-3}^{(k+3)}(x)] \right\} dx$$

$$= \frac{(a_{k+1}^0 a_{k+2}^0)^2}{4\pi} \int_E \frac{f(x)\,\mathrm{Sign}[T'(x)]}{\sqrt{1-T^2(x)}}$$

$$\times \left\{ \frac{2}{a_{k+1}^0} q_1^{(k+3)}(x) + \frac{4}{a_k^0} q_1^{(k+2)}(x) + \frac{2}{a_{k+1}^0} q_1^{(k+1)}(x) \right.$$

$$- \frac{1}{a_{k+1}^0} [q_5^{(k+1)}(x) - q_1^{(k+3)}(x)] - \frac{2}{a_k^0} [q_5^{(k)}(x) - q_1^{(k+2)}(x)]$$

$$\left. - \frac{1}{a_{k+1}^0} [q_5^{(k-1)}(x) - q_1^{(k+1)}(x)] \right\} dx.$$

Since (see [11])

$$q_5^{(m)}(x) = 2T(x)\, q_3^{(m)}(x) - q_1^{(m)}(x) = [4T^2(x) - 1]\, q_1^{(m)}(x),$$

then

$$I_k^{(2)} = \frac{(a_{k+1}^0 a_{k+2}^0)^2}{4\pi} \int_E \frac{f(x)\,\mathrm{Sign}[T'(x)]}{\sqrt{1-T^2(x)}}$$

$$\left\{ \left[\frac{3}{a_{k+1}^0} q_1^{(k+3)}(x) + \frac{6}{a_k^0} q_1^{(k+2)}(x) + \frac{4}{a_{k+1}^0} q_1^{(k+1)}(x) \right. \right.$$

$$\left. + \frac{2}{a_k^0} q_1^{(k)}(x) + \frac{1}{a_{k+1}^0} q_1^{(k-1)}(x) \right]$$

$$\left. - 4T^2(x) \left[\frac{1}{a_{k+1}^0} q_1^{(k+1)}(x) + \frac{2}{a_k^0} q_1^{(k)}(x) + \frac{1}{a_{k+1}^0} q_1^{(k-1)}(x) \right] \right\} dx.$$

It follows from the definition of the class AP_2 and (3.1), (3.2), that

$$q_1^{(m)}(x)=\frac{1}{a_{m+1}^0}(x-b_m^0), \qquad q_1^{(m+2)}(x)=q_1^{(m)}(x).$$

So

$$\frac{1}{a_{k+1}^0}q_1^{(k+1)}(x)+\frac{2}{a_k^0}q_1^{(k)}(x)+\frac{1}{a_{k+1}^0}q_1^{(k-1)}(x)=\frac{2}{a_{k+1}^0 a_{k+2}^0}(2x-b_k^0-b_{k+1}^0)$$

and

$$\frac{3}{a_{k+1}^0}q_1^{(k+3)}(x)+\frac{6}{a_k^0}q_1^{(k+2)}(x)+\frac{4}{a_{k+1}^0}q_1^{(k+1)}(x)$$
$$+\frac{2}{a_k^0}q_1^{(k)}(x)+\frac{1}{a_{k+1}^0}q_1^{(k-1)}(x)=\frac{8}{a_{k+1}^0 a_{k+2}^0}(2x-b_k^0-b_{k+1}^0).$$

Consequently,

$$I_k^{(2)}=\frac{2}{\pi}a_{k+1}^0 a_{k+2}^0\int_E f(x)(2x-b_k^0-b_{k+1}^0)\,\mathrm{Sign}[T'(x)]\sqrt{1-T^2(x)}\,dx.$$

One can calculate the integrand. By (2.1) and (3.1) (for $m=0$) one obtains

$$2T(x)=\frac{1}{a_{k+1}^0 a_{k+2}^0}[(x-b_k^0)(x-b_{k+1}^0)-(a_{k+1}^0)^2-(a_{k+2}^0)^2],$$
$$2T'(x)=\frac{2x-b_k^0-b_{k+1}^0}{a_{k+1}^0 a_{k+2}^0},$$
$$1-T^2(x)=\frac{1}{(2a_{k+1}^0 a_{k+2}^0)^2}\{[(a_{k+1}^0+a_{k+2}^0)^2-(x-b_k^0)(x-b_{k+1}^0)]$$
$$\times[(x-b_k^0)(x-b_{k+1}^0)-(a_{k+1}^0-a_{k+2}^0)^2]\}.$$

Lemma 5 is completely proved.

Corollary 2. *Assume that the recurrence coefficients* $\{a_{n+1}\}$ *and* $\{b_n\}$ *belong to the class* AP_2 *and that they satisfy* (1.3). *Then for every continuous function* f *and for all integers* k *one has*

$$\lim_{n\to\infty}\int_E f(x)\,G_{2n+k}(x)\,d\mu(x)=\frac{2}{\pi}\int_E f(x)\,x^2\sqrt{1-x^2}\,dx.$$

The main result is the following analog of Theorem 1.

THEOREM 2. *Let the Jacobi matrix $J \in AP_2$ and the recursion coefficients a_n, b_n satisfy*

$$\sum_{n=0}^{\infty} (|a_{n+2}-a_n| + |b_{n+2}-b_n|) < \infty \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} (|a_{n+1}-a_n| + |b_{n+1}-b_n|)(|e_n| + |e_{n+1}|) < \infty, \tag{3.7}$$

where $e_n = b_{n-1} + b_n - b_0^0 - b_1^0$, $\lim_{n\to\infty} e_n = 0$. *Then uniformly on every closed subset of* $E_0 := E \setminus \{T^2(x) = 1\}$ *the Trace Formula*

$$\begin{aligned}
&2 \sum_{n=0}^{\infty} \left[(\alpha_{n+2}^2 - \alpha_n^2) + \frac{1}{2}(\beta_{n+1}^2 - \beta_n^2)\right] p_n^2(x) \\
&\quad + \sum_{n=0}^{\infty} [\beta_{n+1}(\gamma_{n+1} - \gamma_n) + 3(\alpha_{n+2}\beta_{n+2} - \alpha_{n+1}\beta_n)]\, p_n(x)\, p_{n+1}(x) \\
&\quad + 2 \sum_{n=0}^{\infty} \alpha_{n+2}(\gamma_{n+2} - \gamma_n)\, p_n(x)\, p_{n+2}(x) \\
&\quad + \sum_{n=0}^{\infty} (\alpha_{n+2}\beta_{n+3} - \alpha_{n+3}\beta_{n+1})\, p_n(x)\, p_{n+3}(x) \\
&= \frac{4}{\pi} (a_1^0 a_2^0)^2 \frac{|T'(x)| \sqrt{1 - T^2(x)}}{w(x)}.
\end{aligned}$$

holds, where α_n, β_n, *and* γ_n *are defined by* (2.4).

Recall, that in our case, the measure μ is absolutely continuous in E_0, and $\mu'(x) = w(x)$ is strictly positive and continuous on E_0 (see [11]).

Proof of Theorem 2. We adapt the methods of [11] to the present situation. It is not difficult to see that from (2.4), (3.6), and (3.7) one obtains

$$\begin{aligned}
&\sum_{n=0}^{\infty} [|\alpha_{n+2}^2 - \alpha_n^2| + |\beta_{n+1}^2 - \beta_n^2| + |\beta_{n+1}(\gamma_{n+1} - \gamma_n)| \\
&\quad + |\alpha_{n+2}\beta_{n+2} - \alpha_{n+1}\beta_n| + |\alpha_{n+2}(\gamma_{n+2} - \gamma_n)| \\
&\quad + |\alpha_{n+2}\beta_{n+3} - \alpha_{n+3}\beta_{n+1}|] < \infty.
\end{aligned}$$

So, the series on the left side of our assertion converges uniformly on the closed subset K from E_0. If we denote its sum by $\psi(x)$, then

$$\lim_{n\to\infty} G_n(x)=\psi(x)$$

uniformly on K; consequently, for every continuous function f and for all integers k one has

$$\lim_{n\to\infty}\int_E f(x)\, G_{2n+k}(x)\, d\mu(x)=\int_E f(x)\,\psi(x)\,w(x)\,dx.$$

On the other hand, using periodicity of $\{a^0_{n+1}\}$ and $\{b^0_n\}$, and by Lemma 5 one obtains

$$\lim_{n\to\infty}\int_E f(x)\, G_{2n+k}(x)\, w(x)\, dx=\frac{4}{\pi}\,(a^0_1 a^0_2)^2\int_E f(x)\,|T'(x)|\,\sqrt{1-T^2(x)}\,dx.$$

This means that for $x\in E_0$

$$\psi(x)\,w(x)=\frac{4}{\pi}\,(a^0_1 a^0_2)^2\,|T'(x)|\,\sqrt{1-T^2(x)}$$

from which the Trace Formula follows, if we use the end of the proof of Lemma 5. This completes the proof of Theorem 2.

COROLLARY 3. *If the recurrence coefficients satisfy*

$$\lim_{n\to\infty} a_n=\frac{1}{2},\qquad \sum_{n=0}^{\infty}|a_{n+2}-a_n|<\infty,\qquad b_n=0\quad (n\in\mathbb{Z}_+),$$

then the Trace Formula

$$\sum_{n=0}^{\infty}(a^2_{n+1}a^2_{n+2}-a^2_{n-1}a^2_n)\,p^2_n(x)$$
$$+\sum_{n=0}^{\infty}a_{n+1}a_{n+2}(a^2_{n+3}+a^2_{n+2}-a^2_{n+1}-a^2_n)\,p_n(x)\,p_{n+2}(x)$$
$$=\frac{1}{\pi}\,\frac{x^2\sqrt{1-x^2}}{w(x)}$$

holds uniformly on every compact subset of E_0.

In fact, in this case

$$\alpha_n=a_{n-1}a_n,\qquad \beta_n=0,\qquad \gamma_n=a^2_n+a^2_{n+1}-(a^0_1)^2-(a^0_2)^2.$$

Remarks. 1. In view of the assumption on the recurrence coefficients the relation $E_0 = (-1, 1)$ holds, and Corollary 3 gives a Trace Formula for the single interval case.

2. Another Trace Formula is obtained in [11, 23].

4. EXAMPLES

1. The sieved Pollaczek polynomials.

(a) The symmetric sieved Pollaczek polynomials [12]. Let $\{C_n^\lambda(x; a; 2)$ $(a>0, \lambda>0)\}$ be the symmetric sieved Pollaczek polynomials of the first kind. Then

$$C_0^\lambda(x; a; 2) = 1,$$

$$C_1^\lambda(x; a; 2) = \frac{x(\lambda + a)}{\lambda},$$

$$xC_{2n+1}^\lambda(x; a; 2) = \frac{1}{2} C_{2n+2}^\lambda(x; a; 2) + \frac{1}{2} C_{2n}^\lambda(x; a; 2),$$

$$xC_{2n}^\lambda(x; a; 2) = \frac{1}{2} \frac{n+2\lambda}{n+a+\lambda} C_{2n+1}^\lambda(x; a; 2) + \frac{1}{2} \frac{n}{n+a+\lambda} C_{2n-1}^\lambda(x; a; 2).$$

If $\{\hat{C}_n^\lambda(x; a; 2)\}$ is the corresponding orthonormal system, then

$$C_n^\lambda(x; a; 2) = \left\{\frac{\pi\Gamma(2\lambda)}{a+\lambda} 2^{-2\lambda+1}\right\}^{1/2} \lambda_n^{1/2} \hat{C}_n^\lambda(x; a; 2)$$

with

$$\lambda_{2n} = \frac{n!\,(a+\lambda)}{(n+a+\lambda)(2\lambda)_n}, \qquad \lambda_{2n+1} = \frac{n!\,(a+\lambda)}{(2\lambda)_{n+1}},$$

where, as usual,

$$(a_n) = \frac{\Gamma(n+a)}{\Gamma(a)}.$$

We have

$$x\hat{C}^{\lambda}_{2n+1}(x;a;2)=\frac{1}{2}\sqrt{\frac{n+1}{n+a+\lambda+1}}\,\hat{C}^{\lambda}_{2n+2}(x;a;2)+\frac{1}{2}\sqrt{\frac{n+2\lambda}{n+a+\lambda}}\,\hat{C}^{\lambda}_{2n}(x;a;2)$$

$$x\hat{C}^{\lambda}_{2n}(x;a;2)=\frac{1}{2}\sqrt{\frac{n+2\lambda}{n+a+\lambda}}\,\hat{C}^{\lambda}_{2n+1}(x;a;2)+\frac{1}{2}\sqrt{\frac{n}{n+a+\lambda}}\,\hat{C}^{\lambda}_{2n-1}(x;a;2).$$

So

$$a_{2n}=\frac{1}{2}\sqrt{\frac{n}{n+a+\lambda}},\qquad a_{2n+1}=\frac{1}{2}\sqrt{\frac{n+2\lambda}{n+a+\lambda}},\qquad b_n=0. \tag{4.1}$$

From (4.1) one easily finds

$$a_{2n}-a_{2n+1}\approx C\frac{1}{n},$$

which means that (1.4) does not hold and hence that Theorem 1 cannot be applied. On the other hand,

$$a_{2(n+1)}-a_{2n}=O\left(\frac{1}{n^2}\right),\qquad a_{2n+1}-a_{2n+3}=O\left(\frac{1}{n^2}\right),$$

which means that Corollary 3 can be applied. Note that when $a=0$ then $C^{\lambda}_{n}(x;a;2)$'s reduce to the $C^{\lambda}_{n}(x;2)$'s of Al-Salam *et al.* [1], for which Theorem 1 cannot be applied, and Corollary 3 can be applied.

(b) The nonsymmetric sieved Pollaczek polynomials ([4]; see also [16]). Given a, b, c, $\lambda\in\mathbb{R}$, we define the 2-sieved 4-parameter Pollaczek polynomials as the characteristic polynomials associated with Jacobi matrix $J(2;a,b,c;\lambda)$, where

$$a_n=\sqrt{\frac{A_{n-1}D_n}{B_{n-1}B_n}},\qquad b_n=\frac{C_n}{D_n}\qquad (n=1,2,...).$$

Here

$$A_{2n}=n+c+2\lambda,\qquad A_{2n+1}=1,\qquad B_{2n}=2(n+a+c+\lambda),\qquad B_{2n+1}=2,$$

$$C_{2n}=-2b,\qquad C_{2n+1}=0,\qquad D_{2n}=n+c+2\lambda-1,\qquad D_{2n+1}=1.$$

For the 4-parameter Pollaczek polynomials see Chihara's book [5, p. 185], whereas for the sieving process see the works [1, 4, 12]. The recurrence coefficients associated with the above orthogonal polynomials

$$a_{2n}=\frac{1}{2}\sqrt{\frac{n+c+2\lambda-1}{n+a+c+\lambda}}, \qquad a_{2n+1}=\frac{1}{2}\sqrt{\frac{n+c+2\lambda}{n+a+c+\lambda}},$$

$$b_{2n}=-\frac{b}{n+a+c+\lambda}, \qquad b_{2n+1}=0$$

are not bounded variation, but it is not difficult to see that the conditions of (3.6) and (3.7) of Theorem 2 are satisfied.

2. Orthonormal polynomial system defined by the recurrence relation (1.1) with

$$a_n=\frac{1}{2}+\frac{(-1)^n D}{n^p}+O\left(\frac{1}{n^{p+1}}\right), \qquad b_n=\frac{(-1)^n R}{n^p}+O\left(\frac{1}{n^{p+1}}\right) \qquad (p\geqslant 1), \tag{4.2}$$

where D and R are constants independent of n, satisfy conditions (3.6) and (3.7).

3. Polynomials orthogonal on two intervals [2, 3, 6, 20]. Denote

$$E_\xi=[-1,-\xi]\cup[\xi,1] \qquad (0\leqslant\xi<1).$$

(a) Let

$$w_0(x)=\sqrt{\frac{x+\xi}{x-\xi}}\sqrt{\frac{1-x}{1+x}} \qquad (x\in E_\xi).$$

In the paper [3] the polynomials

$$\hat{\rho}_n(x;\xi)=\hat{k}_n x^n+\cdots, \qquad \hat{k}_n>0 \quad (n\in\mathbb{Z}_+;\, x\in E_\xi)$$

were introduced; they are orthonormal with respect to the weight $w_0(x)$ on E_ξ.

They satisfy the following three-term recurrence relation

$$a_{n+1}\hat{\rho}_{n+1}(x;\xi)-(x+b_n)\,\hat{\rho}_n(x;\xi)+a_n\hat{\rho}_{n-1}(x;\xi)=0$$

with

$$a_n = \frac{1+(-1)^n \xi}{2} \quad (n=1, 2, ...), \qquad b_n = 0 \quad (n=1, 2, ...), \qquad b_0 = -\frac{1-\xi}{2}.$$

Obviously, the conditions of Theorem 2 are satisfied.

(b) Denote

$$w^{(p,q)}(x) = \begin{cases} \left(\frac{2}{1-\xi^2}\right)^{p+q} (x+\alpha)(x^2-\xi^2)^p (1-x^2)^q \\ \qquad \text{for} \quad -1 \leqslant x \leqslant -\xi \\ -\left(\frac{2}{1-\xi^2}\right)^{p+q} (x+\alpha)(x^2-\xi^2)^p (1-x^2)^q \\ \qquad \text{for} \quad \xi \leqslant x \leqslant 1 \\ 0 \qquad \text{for} \quad x \notin E_\xi, \end{cases}$$

where $p > -1$, $q > -1$, $-\xi \leqslant \alpha \leqslant \xi$.

Let $\{\rho_n^{(p,q)}(x)\}$ $(n \in \mathbb{Z}_+)$ be an orthogonal polynomial system on E_ξ with respect to the weight $w^{(p,q)}(x)$, where the leading coefficients of $\rho_n^{(p,q)}(x)$ are equal to 1. In [2] G. I. Barkov has shown that

$$\rho_{2n}^{(p,q)}(x) = \left(\frac{1-\xi^2}{2}\right)^n \left(\frac{2}{1-\xi^2}\right)^{p+q} I_n^{(p,q)}\left(\frac{2x^2-\xi^2-1}{1-\xi^2}\right),$$

$$\rho_{2n+1}^{(p,q)}(x) = \frac{\rho_{2n+2}^{(p,q)}(x) - m_{2n}\rho_{2n}^{(p,q)}(x)}{x+\alpha},$$

$$m_{2n} = \frac{\rho_{2n+2}^{(p,q)}(\alpha)}{\rho_{2n}^{(p,q)}(\alpha)},$$

where $I_n^{(p,q)}(x)$ are Jacobi polynomials with leading coefficient 1 orthogonal with respect to the weight $(1+x)^p (1-x)^q$ $(-1 \leqslant x \leqslant 1)$. The corresponding orthonormal polynomials $\{\hat{\rho}_n^{(p,q)}\}$ can be represented in the form

$$\hat{\rho}_{2n}^{(p,q)}(x) = (1-\xi^2)^{-(2n+1)/2}\, 2^{-(p+q)/2} \times \left[\frac{n!\, \Gamma(n+p+1)\, \Gamma(n+q+1)\, \Gamma(n+p+q+1)}{\Gamma(2n+p+q+1)\, \Gamma(2n+p+q+2)}\right]^{1/2} \rho_{2n}^{(p,q)}(x)$$

and

$$\hat{\rho}_{2n+1}^{(p,q)}(x) = (1-\xi^2)^{-(2n+1)/2}\, 2^{-(p+q)/2} \times \left[\frac{n!\,\Gamma(n+p+1)\,\Gamma(n+q+1)\,\Gamma(n+p+q+1)}{\Gamma(2n+p+q+1)\,\Gamma(2n+p+q+2)}\right]^{1/2} \times [-m_{2n}]^{-1/2}\, \rho_{2n+1}^{(p,q)}(x),$$

and satisfy the following recurrence relation

$$x\hat{\rho}_n^{(p,q)}(x) = a_{n+1}^{(p,q)}\,\hat{\rho}_{n+1}^{(p,q)}(x) + b_n^{(p,q)}\,\hat{\rho}_n^{(p,q)}(x) + a_n^{(p,q)}\,\hat{\rho}_{n-1}^{(p,q)}(x) \qquad (n \in \mathbb{Z}_+)$$

with

$$a_{2n+1}^{(p,q)} = \sqrt{-m_{2n}},$$
$$a_{2n+2}^{(p,q)} = \frac{1-\xi^2}{\sqrt{-m_{2n}}}\left\{\frac{(n+1)(n+p+1)(n+q+1)(n+p+q+1)}{(2n+p+q+1)(2n+p+q+2)^2\,(2n+p+q+3)}\right\}^{1/2},$$
$$b_n^{(p,q)} = (-1)^n\,\alpha.$$

It is not difficult to see that the corresponding Jacobi matrix belongs to the class AP_2.

In the case $\alpha = \pm\xi$, i.e., $w(x) = |x \pm \xi|\,(x^2-\xi^2)^p\,(1-x^2)^q\,\,(2/(1-\xi^2))^{p+q}$ one obtains

$$\begin{aligned} a_{2n+2}^{(p,q)} &= \sqrt{1-\xi^2}\left[\frac{(n+q+1)(n+1)}{(2n+p+q+3)(2n+p+q+2)}\right]^{1/2} \\ a_{2n+1}^{(p,q)} &= \sqrt{1-\xi^2}\left[\frac{(n+p+q+1)(n+p+1)}{(2n+p+q+1)(2n+p+q+2)}\right]^{1/2} \\ b_n &= (-1)^n\,\xi. \end{aligned} \tag{4.3}$$

These sequences of recurrence coefficients do not satisfy (1.4), but they satisfy the conditions of Theorem 2.

In particular, in the case $\alpha = \xi = 0$, i.e., $w(x) = |x|^{2p+1}\,(1-x^2)^q$ we have the generalized Chebychev polynomials, for which [13, 15] (see (4.2) with $p=1$):

$$a_n^{(p,q)} = \frac{1}{2} + \frac{C_{p,q}}{n}(-1)^n + O\left(\frac{1}{n^2}\right), \qquad b_n = 0.$$

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